JOURNAL OF APPROXIMATION THEORY **46**, 111–118 (1986)

Contributions of Géza Freud to the Theory of Rational Approximation of Functions

VASIL A. POPOV

Bulgarian Academy of Sciences, Institute of Mathematics with Computer Center, 1090 Sofia, Bulgaria

Communicated by Paul G. Nevai

Received June 28, 1984

DEDICATED TO THE MEMORY OF GÉZA FREUD

Although the articles of G. Freud in the rational approximation of functions are not numerous, they have had a remarkable impact on the contemporary state of affairs of this theory.

I would like to emphasize that in the development of the theory of rational approximation of functions there is a point of discontinuity, a jump in 1964, when D. J. Newman [1] proved that the best uniform rational approximation of order n of the function $|x|$ in the interval $[-1, 1]$ has order $O(e^{-c\sqrt{n}})$. Compare this with the classical result of S. Bernstein according to which the order of the best uniform approximation of $|x|$ in the interval $[-1, 1]$ by means of polynomials of degree n is only $O(n^{-1})$.

After Newman's result the interest in rational approximations increased rapidly; many authors found classes of functions for which the order of the best uniform (or L_p) rational approximation is better than the order of the best uniform (or L_p) polynomial approximation (with the same number of parameters). First, P. Turán and P. Szüsz [2–4] found such classes of functions, namely, the functions which are piecewise analytic, the convex Lip 1 functions, and the functions with derivatives which are convex and Lip 1. This type of problem has its subsequent refinement and generalization in Freud's works in rational approximation.

Let us make some notations first. We denote by $\|f\|_{C[a,b]}$ the supremum (uniform) norm of the function f in the interval $[a, b]$, i.e.,

$$\|f\|_{C[a,b]} = \sup\{|f(x)|: x \in [a, b]\}.$$

0021-9045/86 $3.00

Let P_n be the set of all algebraic polynomials of degree n, R_n, the set of all rational functions of order n, i.e., $r_n \in R_n$ if

$$r_n(x) = \frac{a_m x^m + \cdots + a_0}{b_k x^k + \cdots + b_0}, \qquad m \leqslant n, k \leqslant n.$$

We denote by $E_n(f; [a, b])$ the best uniform approximation of the function f by means of algebraic polynomials of degree n in the interval $[a, b]$,

$$E_n(f; [a, b]) = \inf\{\| f - p \|_{C[a,b]} : p \in P_n\}.$$

Let $R_n(f; [a, b])$ be the best uniform approximation of the function f in the interval $[a, b]$ by means of rational functions of order n

$$R_n(f; [a, b]) = \inf\{\| f - r \|_{C[a,b]} : r \in R_n\}.$$

We will need the modulus of continuity of the function f

$$\omega(f; \delta) = \sup\{| f(x) - f(x') | : | x - x' | \leqslant \delta\}.$$

Then $f \in \text{Lip}\, \alpha$ iff $\omega(f; \delta) = O(\delta^\alpha)$. We will also use the second modulus of continuity (modulus of smoothness) of the function f,

$$\omega_2(f; \delta) = \sup\{| f(x+h) - 2f(x) + f(x-h) | : x+h, x-h \in [a, b], |h| \leqslant \delta\}.$$

Then f belongs to the Zygmund class iff $\omega_2(f; \delta) = O(\delta)$.

1. Functions with Derivatives of Bounded Variation

The results of Turán and Szüsz can be written as follows: If f is convex and Lip 1 function in the interval $[a, b]$ then [2]

$$R_n(f; [a, b]) = O\left(\frac{\log^4 n}{n^2}\right).$$

If $f^{(k-1)}$ is convex and Lip 1 function in the interval $[a, b]$ then [3]

$$R_n(f; [a, b]) = O\left(\frac{\log^{2k+2} n}{n^{k+1}}\right).$$

In his remarkable work [5] using a new method Freud improves these results as follows:

If $f^{(k)}$ is of bounded variation in the interval $[a, b]$ and $k \geqslant 1$ then

$$R_n(f; [a, b]) = O\left(\frac{\log^2 n}{n^{k+1}}\right) \tag{1}$$

that is $\log^{2k+2} n$ is replaced by $\log^2 n$. However, this is not the most essential in Freud's work. Turán and Szüsz use intermediate piecewise linear approximation of the function f (or $f^{(k-1)}$) and afterwards they approximate the piecewise linear function by means of rational function using Newman's result. Freud uses the best polynomial approximation of f of order s in each subinterval $[x_{i-1}, x_i]$, $i = 1,..., m$, $a = x_0 < x_1 < \cdots x_m = b$, in a suitably chosen partition of the interval $[a, b]$. After that the "norms" these polynomials using the Chebyshev poynomials of order s and he connects these rational functions of order s by means of rational approximation of the jump. All this is done very technically.

In 1976, using a modification of Freud's method (in which instead of the best polynomial approximation the best rational approximation is used), I obtained the final result [16]:

If $f^{(k)}$ is of bounded variation in the interval $[a, b]$, then

$$R_n(f; [a, b]) \leqslant c(k) \frac{(b-a)^k V_a^b f^{(k)}}{n^{k+1}}, \tag{2}$$

where $V_a^b \phi$ denotes the variation of the function ϕ in the interval $[a, b]$ and $c(k)$ is a constant, depending only on k.

In his work [6] Freud proved that it is not possible to obtain estimates better than (2) for functions the kth derivative of which is of bounded variation.

As G. Freud remarks in [7], (2) implies the famous Newman's conjecture according to which if $f \in \text{Lip}\ 1$ then $R_n(f; [a, b]) = o(n^{-1})$.

Freud's method in [7] is a general method for obtaining estimates of the type $o(n^{-\gamma})$ (also see [15]). In fact, using Freud's method and (2) it is not very difficult to prove that we have $R_n(f; [a, b]) = o(n^{-1})$ not only when $f \in \text{Lip}\ 1$, but also when f is absolutely continuous and belongs to the Zygmund class ($\omega_2(f; \delta) = O(\delta)$).

Let us present here Freud's method. Let f be absolutely continuous function on the interval $[0, 1]$. Then the piecewise linear function g_m: $g_m(i/m) = f(i/m)$, $i = 0,..., m$, linear in the intervals $[(i-1)/m, i/m]$, $i = 1,..., m$, satisfies the inequality

$$\| f - g_m \|_{C[a,b]} < \omega_2\left(f; \frac{1}{m}\right) \tag{3}$$

(by Whitney's theorem, see [7, 15]).

On the other hand

$$g'_m(x) = m(f(i/m) - f((i-1)/m)) \qquad \text{for} \quad x((i-1)/m, i/m),$$

and therefore the variation $V_0^1 g'_m$ of g'_m is

$$\begin{aligned} V_0^1 g'_m &= m \sum_{i=1}^{m-1} \left| f\left(\frac{i+1}{m}\right) - 2f\left(\frac{i}{m}\right) + f\left(\frac{i-1}{m}\right) \right| \\ &= m \sum_{i=1}^{m-1} \left| \int_{(i-1)/m}^{i/m} (f'(t+1/m) - f'(t))\, dt \right| \\ &\leqslant m \sum_{i=1}^{m-1} \int_{(i-1)/m}^{i/m} |f'(t+1/m) - f'(t)|\, dt \\ &= m \int_0^{1-1/m} |f'(t+1/m) - f'(t)|\, dt \leqslant m\omega(f'; 1/m)_L, \end{aligned} \tag{4}$$

where

$$\omega(\phi; \delta)_L = \sup_{0 < h \leqslant \delta} \int_0^{1-h} |\phi(t+h) - \phi(t)|\, dt$$

is the integral modulus of continuity.

Now if we have (2) for $k=1$ then we obtain from (3) and (4)

$$\begin{aligned} R_n(f; [0, 1]) &\leqslant \|f - g_m\|_{C[0,1]} + R_n(g_m; [0, 1]) \\ &\leqslant \omega_2(f; 1/m) + c(1)\, m\omega(f'; 1/m)_L\, n^{-2}. \end{aligned}$$

Since f' is integrable, $\omega(f'; 1/m)_L \to 0$ $(m \to \infty)$. If $\omega_2(f; \delta) = O(\delta)$ then setting $m = n/\sqrt{\omega(f'; n^{-1})_L}$ we obtain

$$R_n(f; [0, 1]) = O(\sqrt{\omega(f'; n^{-1})_L}/n),$$

that is, if f is absolutely continuous and belongs to the Zygmund class then $R_n(f; [0, 1]) = o(1/n)$.

2. Functions of Bounded Variation with a Given Modulus of Continuity

Now let us consider other classes of functions for which Freud obtained that the best rational approximation is better than the corresponding best polynomial approximation: the classes of functions of bounded variation with a given modulus of continuity $\omega(\delta)$ of the type $\omega(\delta) = O(\delta^\alpha)$ or $\omega(\delta) = O(\log^{-\gamma} 1/\delta)$.

Using his method from [5], Freud obtains the following results:

If f is a function with bounded variation on the interval $[a, b]$ and $\omega(f; \delta) = O(\delta^\alpha)$ then

$$R_n(f; [a, b]) = O\left(\frac{\log^2 n}{n}\right).$$

If f is a function with bounded variation on the interval $[a, b]$ and $\omega(f; \delta) = O(\log^{-\gamma} 1/\delta)$ then

$$R_n(f; [a, b]) = O(n^{-\gamma/(2+\gamma)}).$$

After Freud, many mathematicians worked in this area, e.g., E. P. Delgenko, A. A. Abdulgaparov, A. P. Bulanov, P. Petrushev. The final result belongs to P. Petrushev and is the following.

If f is a function with bounded variation on the interval $[a, b]$ and $\omega(f; \delta) = O(\delta^\alpha)$ then [19]

$$R_n(f; [a, b]) = O\left(\frac{\log n}{n}\right). \qquad (5)$$

If f is a function with bounded variation on the interval $[a, b]$ and $\omega(f; \delta) = O(\log^{-\gamma} 1/\delta)$ then [19]

$$R_n(f; [a, b]) = O(n^{-\gamma/(1+\gamma)}).$$

The fact that these estimates are sharp follows from Bulanov's and Petrushev's results [20]. The interesting point is that in formula (5) the $\log n$ cannot be omitted.

3. Localization Theorems

In the same work [5] Freud also gives the first variant of the so-called "localization theorem." Roughly speaking this means the following.

Let $a = x_0 < x_1 < \cdots < x_m = b$ be a partition of the interval $[a, b]$ and let f be such that there exist algebraic polynomials $p_\nu^{(i)}$ of degree ν such that

$$\| f - p_\nu^{(i)}) \|_{C[x_{i-1}, x_i]} \leqslant \varepsilon_\nu, \qquad i = 1, ..., m.$$

Then

$$R_n(f; [a, b]) \leqslant c(f)(\varepsilon_{\nu_n} + e^{-1/2\sqrt{\nu_n}}),\ \nu_n = 2\left[\frac{n}{4m}\right].$$

From this "weak localization theorem" it is possible to obtain many consequences, for example Turán's results.

Let us mention that this theorem has a "strong" version (see Szabados [14]) where the piecewise polynomial approximation is replaced by piecewise rational approximation (with additional assumption on the order of the derivatives of the approximating functions).

Another generalization is when the number m of partitions increases simultaneously with n, see [10]. As an application of this theorem, Freud and Szabados in the same paper proved an estimation for the rational approximation to x^α, $0<\alpha<1$, $0\leqslant x\leqslant 1$ and to $x^{p/q}$, $0<x<\infty$, p, q-integers. The result for x^α is the following:

$$R_n(x^\alpha; [0, 1]) = O(e^{-c(\alpha)\sqrt[3]{n}}), \qquad 0<\alpha<1.$$

Let us mention that this result has been improved by Gončar [17] to the final result:

$$R_n(x^\alpha; [0, 1]) = O(e^{-c(\alpha)\sqrt{n}}).$$

In his work [8] Freud discusses one conjecture of P. Turán: let $a=x_0<x_1<\cdots<x_m=b$ be a partition of the interval $[a, b]$ and let f be a function, belonging to Lip α in each inner point of (x_{i-1}, x_i) and to Lip β at the points x_i, $i=0,..., m$, $\beta<\alpha$; then

$$R_n(f; [a, b]) = O(n^{-\alpha}).$$

Freud gives the following theorem:

Let $\omega(\delta)$ be a modulus of continuity, $a=x_0<x_1<\cdots<x_m=$ be a partition of the interval $[a, b]$, $\eta_i=\frac{1}{2}(x_{i-1}+x_i)$ and let f be such that

$$|f(y_1)-f(y_2)| \leqslant K\omega(|\sqrt{y_1-x_i}-\sqrt{y_2-x_i}|)$$

if y_1, y_2 both belong to $[\eta_i, x_i]$ or $[x_i, \eta_{i+1}]$, $i=1,..., m-1$. Then

$$R_n(f; [a, b]) = O(\omega(1/n)).$$

As a consequence of this theorem G. Freud obtains Turán's conjecture for $\alpha/2\leqslant\beta\leqslant\alpha$.

4. Infinite Interval

Let us go now to another essential theme in Freud's works in rational approximation which consists of rational approximation on the whole real line.

In his work [9] Freud gives the following analogue to Newman's result:

For every $n>4$ there exists a rational function $r_n \in R_n$ such that for every x we have

$$\frac{1}{1+x^2}\,||x|-r_n(x)| \leqslant \frac{3}{2}\, e^{-\sqrt{(n-4)/2}}.$$

On the other hand, there does not exist a sequence of rational functions such that

$$\frac{1}{1+x^2}\,||x|-r_n(x)| \leqslant \frac{1}{4}\, e^{-9\sqrt{n}}.$$

This means that the rational approximation of $|x|$ with a weight $1/(1+x^2)$ on the whole real line has order $O(e^{-c\sqrt{n}})$.

For weighted rational approximation this result gives corollaries which are similar to the corollaries of Newman's result for finite interval. Let us mention some of them.

In [12] Freud and Szabados obtain an analogue of the "weak localization theorem" for the infinite interval. As a consequence they deduct the following result.

For every natural $n>3$ there exists $r_n \in R_n$ such that

$$|\arctan x - r_n(x)| \leqslant \frac{|x|^3}{1+x^2}\left(\frac{|x|}{1+1+x^2}\right)^{n-3}$$

The localization theorem of Freud has further its development in the work [18].

Acknowledgments

Freud presented some of his results at the International Congress of Mathematicians, Nice, 1970 [13]. Finally, I want to point out that it was Geza Freud who aroused my interest in the theory of rational approximation, a fascinating field of Approximation Theory.

References

1. D. J. Newman, Rational approximation to $|x|$, *Michigan Math. J.* **11** (1) (1964), 11–14.
2. P. Szüsz and P. Turán, On the constructive theory of functions II, *Studia Sci. Math. Hungar.* **1** (1966), 65–69.
3. P. Szüsz and P. Turán, A Konstruktiv Függvénytan egy újabb irányáról, MTA III, **16** (1966), 33–45.
4. P. Szüsz and P. Turán, On the construktive Theory of functions III, *Studia Sci. Math. Hungar.* **1** (1966), 315–322.

5. G. FREUD, Über die Approximation reeler Functionen durch rationale gebrochene Functionen, *Acta Math. Acad. Sci. Hungar.* **17** (1966), 313–324.
6. G. FREUD, On the rational approximation of differentiable functions, *Studia Sci. Math. Hungar.* **5** (1970), 437–439.
7. G. FREUD, On rational approximation of absolutely continuous functions, *Studia Sci. Math. Hungar.* **3** (1968), 383–386.
8. G. FREUD, A contribution to the problem of rational approximation of real functions, *Studia Sci. Math. Hungar.* **2** (1967), 419–423.
9. G. FREUD, A remark concerning the rational approximation of $|x|$, *Studia Sci. Math. Hungar.* **2** (1967), 115–118.
10. G. FREUD AND J. SZABADOS, Rational approximation to x^{α}, *Acta Math. Acad. Sci. Hungar.* **18** (1967), 393–399.
11. G. FREUD AND J. SZABADOS, On rational approximation, *Studia Sci. Math. Hungar.* **2** (1967), 215–219.
12. G. FREUD AND J. SZABADOS, Rational approximation on th whole real axis, *Studia Sci. Math. Hungar* **3** (1968), 201–209.
13. G. FREUD, "On Rational Approximation of Real Continuous Functions," Congress Intern, des Matématiciens, s. 126, Nice, 1970.
14. J. SZABADOS, Generalization of two theorems of G. Freud concerning rational approximation, *Studia Sci. Math. Hungar.* **2** (1967), 73–80.
15. G. FREUD AND V. A. POPOV, Some questions connected with approximation by splines and polynomials, *Studia Sci. Math. Hungar.* **5** (1970), 161–171.
16. V. A. POPOV, Uniform rational approximation of the class V_r and its applications, *Acta Math. Acad. Sci. Hungar.* **29** (1977), 119–129.
17. A. A. GONČAR, Estimates of the growth of rational functions and some applications, *Math. Sb.* **72** (114) (1967), 489–503.
18. V. A. POPOV AND J. SZABADOS, On a general localization theorem and some applications in the theory of rational approximation, *Acta Math. Acad. Sci. Hungar.* **25** (1974), 165–170.
19. P. PETRUSHEV, Uniform rational approximaion of functions with bounded variation, PLISKA *Math. Stud. Bulgar.* **1** (1977), 145–155.
20. A. P. BULANOV, Rational Approximation of continuous functions with bounded variation. *Izv. Akad. Nauk. USSR, Ser. Math.* **39** (1975), 1142–1180.